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Non-standard quantum $(1+1)$ Poincaré group: a T-matrix approach

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Abstract. The Hopf algebra dual form for the non-standard uniparametric deformation of the $(1+1)$ Poincaré algebra $iso(1,1)$ is deduced. In this framework, the quantum coordinates that generate $Fun_w(ISO(1,1))$ define an infinite dimensional Lie algebra. The T-matrix formalism is used to derive a universal R-matrix for both $U_w iso(1,1)$ and $Fun_w(ISO(1,1))$. It is also shown how these results can be generalized for the triangular deformations of $(1+1)$ Poincaré and Galilei algebras that include a spacetime dilation generator.

1. Introduction

Quantum deformations were introduced as quantizations of some integrable models characterized by quadratic Poisson brackets [1]. The essential relation between these brackets and Lie bialgebras was established by Drinfel'd [2]. This link provides a framework in which to understand the quantization of such Poisson–Lie structures as a dual process to the (bialgebra) deformation of universal enveloping Lie algebras [3]. More recently, the introduction of the Hopf algebra dual form T has summarized duality in a universal (representation independent) setting [4–6]. The transfer matrices of certain integrable systems can be seen as particular realizations of the dual form T and, conversely, the construction of new integrable models could be guided by the obtention of new T-matrices. In general, this approach provides a 'canonical' formalism in which quantum objects are defined in a completely similar language to their classical counterparts. This 'proper' setting provides a generalization of many group theoretical results to the non-commutative cases in a straightforward—although possibly cumbersome—way (for instance, see [7] for an example in the context of q-special function theory). From an algebraic point of view, the T-matrix approach emphasizes the equivalent role that both quantum algebras and quantum groups play as Hopf algebra deformations and reveals the importance of solvable Lie algebras in this context, a fact that has also been pointed out in [8].

The Lie bialgebra structures compatible with a determined Lie algebra give a first-order characterization for its quantum algebra deformations. So far, the deformations coming from the (non-degenerate) coboundary Lie bialgebra structures classified by Belavin and Drinfel'd [9] have been deeply studied (the so-called 'standard' deformations [10, 11]). In some cases, their corresponding T-matrices have been deduced and, by using contraction methods, these results have been extended to some quantum non-semisimple groups [4–6, 12, 13].

In this paper we deal with the inhomogeneous algebra $iso(1,1) \simeq t_2 \odot so(1,1)$ with classical commutation rules

$$[K, P_\pm] = \pm 2P_\pm \qquad [P_+, P_-] = 0. \tag{1.1}$$

As a real form, the algebra $iso(1,1)$ is isomorphic to the $(1+1)$-dimensional Poincaré algebra: K and $P_\pm$ generate, respectively, the boosts and the translations along the light-cone. A non-standard coboundary Lie bialgebra $(iso(1,1), \delta^{(\mathrm{n})})$ is generated by the classical r-matrix

$$r^{(\mathrm{n})} = K \wedge P_+. \tag{1.2}$$

As usual, $\delta^{(\mathrm{n})}(X) = [1 \otimes X + X \otimes 1, r^{(\mathrm{n})}]$ and $r^{(\mathrm{n})}$ verifies the classical Yang–Baxter equation (CYBE). The set of coboundary structures for this algebra is completed by the Lie bialgebra $(iso(1,1), \delta^{(\mathrm{s})})$ given by

$$r^{(\mathrm{s})} = K \wedge (P_- + P_+) \tag{1.3}$$

that generates the standard deformation [14, 15] ($r^{(\mathrm{s})}$ fulfils the modified CYBE). As a particular feature of the $(1+1)$ Poincaré algebra [16], there also exists a non-coboundary $iso(1,1)$ bialgebra with cocommutator

$$\delta^{(\mathrm{nc})}(K) = 0 \qquad \delta^{(\mathrm{nc})}(P_\pm) = P_\pm \wedge K. \tag{1.4}$$

It can be easily checked that the quantum Poincaré algebra of [17] is a deformation of $iso(1,1)$ 'in the direction' of $\delta^{(\mathrm{nc})}$. These three Lie bialgebras were characterized in [18].

Starting from the quantization $U_w iso(1,1)$ (reviewed in section 2) of the Lie bialgebra $(iso(1,1), \delta^{(\mathrm{n})})$, the construction of its Hopf algebra dual form is developed in section 3. The main result of this section is that the non-standard quantum T-matrix is written as a product of the usual exponentials. We recall that the $(1+1)$ Poincaré dual form given in [12] (that corresponds to the quantum algebra [17]) does contain q-exponential factors.

In section 4, the non-standard quantum Poincaré group $Fun_w(ISO(1,1))$ is deduced. We emphasize a relevant difference of the result so obtained with respect to the T-matrices already known: in [4, 5, 12, 13] the classical dynamical variables (i.e. the group coordinates under a certain factorization of the group elements) generate a (solvable) finite dimensional Lie bialgebra (g_x, δ), and the quantum group is constructed as a deformation $U_q g_x$. In our case, the Poisson algebra g_x defined by the Sklyanin bracket coming from (1.2) is an infinite dimensional Lie algebra, and its quantization is obtained by applying the Weyl rule. In particular, this enhances the differences between the T-matrix of [12] and the one presented here.

Further applications of the T-matrix are developed in section 5. First, the dual form can be used to obtain a transformation of the light-cone quantum coordinates in terms of the quantum spacetime ones at the level of the quantum universal enveloping algebra and in such a way that duality is preserved. Second, a universal R-matrix for $U_w iso(1,1)$ is derived by making use of the existence of an algebra homomorphism and coalgebra antihomomorphism that carries the quantum group $Fun_w(ISO(1,1))$ into the quantum algebra $U_w iso(1,1)$. Finally, and starting from these results, the Hopf algebra dual form T and a universal R-matrix are easily constructed for a quantum $(1+1)$ Poincaré algebra enlarged with a dilation. A contraction of this structure gives rise to a $(1+1)$ quantum Galilei algebra, which can be identified in the context of the null-plane deformation of the $(2+1)$ Poincaré algebra given in [19] with the isotopy (Hopf) subalgebra of the null-plane.

2. Non-standard quantum $iso(1,1)$ algebra

Up to equivalence, there exist three types of non-trivial $sl(2,\mathbb{R})$ (coboundary) Lie bialgebra structures, as associated with the three types of orbits of $SL(2,\mathbb{R})$ acting on $sl(2,\mathbb{R})$ via the adjoint action. In this case $\wedge^2 sl(2,\mathbb{R})$ is isomorphic to the adjoint representation of $sl(2,\mathbb{R})$ and therefore bivectors can be identified to vectors, which are space-like, time-like and light-like. One Lie bialgebra is generated by $r^{(s)} = \lambda J_+ \wedge J_-$ ($\lambda \in \mathbb{R}$) and underlies the (standard) Drinfel'd–Jimbo deformation for this algebra. The non-standard bialgebra is related to $r^{(n)} = J_3 \wedge J_+$ and its quantization was developed in [20].

At a classical level, the involutive automorphism of $sl(2,\mathbb{R})$ given by $S(J_3, J_\pm) = (J_3, -J_\pm)$ induces an Inönü–Wigner contraction of this algebra that is obtained as the limit $\varepsilon \to 0$ of the transformation $(K, P_\pm) := \Gamma(J_3, J_\pm) = (J_3, \varepsilon J_\pm)$. The algebra that arises under such a contraction is just (1.1). This procedure can be extended to the quantum case by taking into account the following 'quantum' automorphism of $U_z sl(2,\mathbb{R})$

$$S_q(J_3, J_\pm; z) = (J_3, -J_\pm; -z). \tag{2.1}$$

The involution S_q gives rise to a generalized Inönü–Wigner contraction:

$$(K, P_\pm; w) := \Gamma_q(J_3, J_\pm; z) = (J_3, \varepsilon J_\pm; z/\varepsilon) \tag{2.2}$$

where K and $P_\pm$ and w are the Lie generators and the deformation parameter of the contracted quantum algebra $U_w iso(1,1)$. By applying the transformation Γ_q onto the deformation given in [20] and making the limit $\varepsilon \to 0$, the quantum algebra $U_w iso(1,1)$ is obtained (see [19]):

$$\begin{aligned} \Delta P_+ &= 1 \otimes P_+ + P_+ \otimes 1 \\ \Delta P_- &= \mathrm{e}^{-wP_+} \otimes P_- + P_- \otimes \mathrm{e}^{wP_+} \\ \Delta K &= \mathrm{e}^{-wP_+} \otimes K + K \otimes \mathrm{e}^{wP_+} \end{aligned} \tag{2.3}$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -\mathrm{e}^{wP_+} X \mathrm{e}^{-wP_+} \qquad \text{for } X \in \{K, P_\pm\} \tag{2.4}$$

$$[K, P_+] = 2\frac{\sinh(wP_+)}{w} \qquad [K, P_-] = -2P_- \cosh(wP_+) \qquad [P_+, P_-] = 0. \tag{2.5}$$

This structure is a quantization of the non-standard Lie bialgebra of $iso(1,1)$ generated by $r^{(n)} = K \wedge P_+$. It is worth recalling that this quantum algebra was first discovered in [21] with no reference to contraction procedures. The centre of $U_w iso(1,1)$ is generated by

$$C_w = 2P_- \frac{\sinh(wP_+)}{w}. \tag{2.6}$$

A classical 3×3 matrix representation of $iso(1,1)$ is given by

$$D(K) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} \qquad D(P_+) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad D(P_-) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.7}$$

and it can be easily checked that this is also a representation of the quantum commutation rules (2.5) (note that $D(P_+)$ is an idempotent matrix).

3. The universal T-matrix

Let us consider the Hopf algebra dual form [4, 5]

$$T = \sum_{\mu} X^{\mu} \otimes p_{\mu} \tag{3.1}$$

where X^{μ} runs over all the elements of the basis of a given Hopf algebra and p_{μ} are the elements of the dual basis (summation over repeated indices will be omitted from now on).

In this section we give the construction of the universal T-matrix (3.1) for the Hopf algebra $U_w iso(1,1)$. A suitable basis X^{μ} to work turns out to be $X^{abc} = A_-^a A_+{}^b A^c$ (hereafter this ordering will be preserved), where

$$A_+ = P_+ \qquad A_- = \mathrm{e}^{-wP_+} P_- \qquad A = \mathrm{e}^{wP_+} K. \tag{3.2}$$

The following coproduct and commutation rules are deduced from (2.3) and (2.5):

$$\begin{aligned} \Delta(A_+) &= 1 \otimes A_+ + A_+ \otimes 1 \\ \Delta(A_-) &= \mathrm{e}^{-2wA_+} \otimes A_- + A_- \otimes 1 \\ \Delta(A) &= 1 \otimes A + A \otimes \mathrm{e}^{2wA_+} \end{aligned} \tag{3.3}$$

$$[A, A_+] = \frac{\mathrm{e}^{2wA_+} - 1}{w} \qquad [A, A_-] = -2A_- \mathrm{e}^{2wA_+} \qquad [A_+, A_-] = 0. \tag{3.4}$$

The dual basis to X^{abc} is defined by

$$\langle p_{abc}, X^{lmn} \rangle = \delta_a^l \, \delta_b^m \, \delta_c^n \tag{3.5}$$

and we set $\hat{a}_- = p_{100}$, $\hat{a}_+ = p_{010}$ and $\hat{\chi} = p_{001}$.

Theorem 3.1. The dual basis p_{qrs} can be expressed in terms of the dual coordinates $\hat{a}_-, \hat{a}_+, \hat{\chi}$ in the form

$$p_{qrs} = \frac{\hat{a}_-^q}{q!} \frac{\hat{a}_+^r}{r!} \frac{\hat{\chi}^s}{s!} \tag{3.6}$$

and the Hopf algebra dual form T reads

$$T = \mathrm{e}^{A_- \otimes \hat{a}_-} \mathrm{e}^{A_+ \otimes \hat{a}_+} \mathrm{e}^{A \otimes \hat{\chi}}. \tag{3.7}$$

The proof of this statement follows by considering the structure tensor F that gives us the coproduct of an arbitrary element of $U_w iso(1,1)$

$$\Delta(X^{abc}) := F^{abc}_{lmn;qrs} \, X^{lmn} \otimes X^{qrs} \tag{3.8}$$

and, by means of duality, the product in $Fun_w(ISO(1,1))$

$$p_{lmn} \, p_{qrs} = F^{abc}_{lmn;qrs} \, p_{abc}. \tag{3.9}$$

In particular, it is easy to see that

$$\begin{aligned} F^{abc}_{000;qrs} &= \delta_q^a \, \delta_r^b \, \delta_s^c \\ F^{abc}_{lmn;000} &= \delta_l^a \, \delta_m^b \, \delta_n^c \\ F^{000}_{lmn;qrs} &= \delta_l^0 \, \delta_m^0 \, \delta_n^0 \, \delta_q^0 \, \delta_r^0 \, \delta_s^0. \end{aligned} \tag{3.10}$$

Two recurrence relations for F can be deduced by observing that

$$\Delta(X^{abc}) = \Delta(A_-)\Delta(X^{(a-1)bc}) \tag{3.11}$$

$$\Delta(X^{abc}) = \Delta(A_+)\Delta(X^{a(b-1)c}) \tag{3.12}$$

and using (3.3) and (3.4). In this way we find

$$F^{abc}_{lmn;qrs} = \sum_{k=0}^{m} F^{(a-1)bc}_{lkn;(q-1)rs} \frac{(-2w)^{m-k}}{(m-k)!} + F^{(a-1)bc}_{(l-1)mn;qrs} \qquad a \geqslant 1 \tag{3.13}$$

$$F^{abc}_{lmn;qrs} = F^{a(b-1)c}_{l(m-1)n;qrs} + F^{a(b-1)c}_{lmn;q(r-1)s} \qquad b \geqslant 1 \tag{3.14}$$

where it is assumed that the corresponding component of F vanishes if any of the indices q, l, m, r takes negative values.

The recurrence relation corresponding to $\Delta(X^{abc}) = \Delta(X^{ab(c-1)})\Delta(A)$ is much harder to find in general. However, for our purposes, we shall only need some particular cases of it. Relations (3.13) and (3.14), together with (3.10), lead to

$$F^{abc}_{100;qrs} = a\,\delta^a_{q+1}\,\delta^b_r\,\delta^c_s \qquad a \geqslant 1 \tag{3.15}$$

$$F^{abc}_{010;qrs} = b\,\delta^a_q\,\delta^b_{r+1}\,\delta^c_s \qquad b \geqslant 1 \tag{3.16}$$

and, by simple considerations, we can also find that

$$F^{abc}_{lmn;001} = c\,\delta^a_l\,\delta^b_m\,\delta^c_{n+1} \qquad c \geqslant 1. \tag{3.17}$$

These are the elements of F that are relevant in order to compute the dual basis. Now, with the aid of (3.9) and (3.15), the following relation holds

$$p_{100}\,p_{(q-1)rs} = F^{abc}_{100;(q-1)rs}\,p_{abc} = a\,\delta^a_q\,\delta^b_r\,\delta^c_s\,p_{abc} = q\,p_{qrs} \tag{3.18}$$

hence,

$$p_{qrs} = \frac{\hat{a}_-}{q}\,p_{(q-1)rs} = \cdots = \frac{\hat{a}_-^q}{q!}\,p_{0rs}. \tag{3.19}$$

Straightforward calculations based on (3.16) and (3.17) together with the fact that $p_{000} = 1$ complete the proof of the theorem.

4. The quantum group $Fun_w(ISO(1,1))$

The structure tensor F also allows us to deduce the commutation rules between the generators of $Fun_w(ISO(1,1))$ which are given in the next proposition. Note that these relations have already been obtained in [18].

Proposition 4.1. The algebra $Fun_w(ISO(1,1))$ is the algebra of functions on $\hat{a}_-$, $\hat{a}_+$ and $\hat{\chi}$ modulo the relations

$$[\hat{\chi}, \hat{a}_+] = w(e^{2\hat{\chi}} - 1) \qquad [\hat{\chi}, \hat{a}_-] = 0 \qquad [\hat{a}_+, \hat{a}_-] = -2w\,\hat{a}_-. \tag{4.1}$$

Proof. The commutation relation of any two elements of $Fun_w(ISO(1,1))$ is

$$[p_{lmn}, p_{qrs}] = (F^{abc}_{lmn;qrs} - F^{abc}_{qrs;lmn})\,p_{abc}. \tag{4.2}$$

The explicit expressions (4.1) for $[\hat{\chi}, \hat{a}_-] \equiv [p_{001}, p_{100}]$ and $[\hat{a}_+, \hat{a}_-] \equiv [p_{010}, p_{100}]$ are straightforwardly derived from the relations involving F in the previous section. In spite of the absence of a third general recurrence relation, we can again find the particular values of F involved in

$$[\hat{\chi}, \hat{a}_+] = (F^{abc}_{001;010} - F^{abc}_{010;001})\,p_{abc}. \tag{4.3}$$

From (3.16) we obtain that $F^{abc}_{010;001} = b\,\delta^a_0\,\delta^b_1\,\delta^c_1$. On the other hand, $F^{abc}_{001;010}$ gives the coefficient of the term $A \otimes A_+$ in $\Delta(X^{abc})$. Such a term appears either when $b = c = 1$

(this contribution annihilates the previous p_{011} term) or in the cases $a = b = 0$ and c arbitrary (note that, since $[A, A_{\pm}]$ does not produce A, this generator cannot appear as a byproduct of reordering processes). As a consequence,

$$F^{abc}_{001;010}\, p_{abc} = p_{011} + 2w\, p_{001} + 4w\, p_{002} + \cdots + 2^k w\, p_{00k} + \cdots. \tag{4.4}$$

Therefore,

$$[\hat{\chi}, \hat{a}_+] = w \sum_{k=1}^{\infty} \frac{1}{k!} 2^k \hat{\chi}^k = w(\mathrm{e}^{2\hat{\chi}} - 1). \tag{4.5}$$

Note that two algebras (4.1) with parameters w and w' (both of them different from zero) are isomorphic.

Since the classical fundamental representation of $iso(1,1)$ (2.7) is a fundamental one for the quantum algebra $U_w iso(1,1)$, the specialization of the T-matrix (3.7) to this realization turns out to be formally identical to the classical $ISO(1,1)$ group element, but now with non-commutative entries:

$$T^D = \mathrm{e}^{D(A_-)\hat{a}_-} \mathrm{e}^{D(A_+)\hat{a}_+} \mathrm{e}^{D(A)\hat{\chi}} = \begin{pmatrix} 1 & 0 & 0 \\ \hat{a}_- + \hat{a}_+ & \cosh 2\hat{\chi} & -\sinh 2\hat{\chi} \\ \hat{a}_- - \hat{a}_+ & -\sinh 2\hat{\chi} & \cosh 2\hat{\chi} \end{pmatrix}. \tag{4.6}$$

Thus, the multiplicative property of T leads to the coproduct for $Fun_w(ISO(1,1))$ (note that $D(P_{\pm}) = D(A_{\pm})$, $D(K) = D(A)$):

$$\begin{aligned} &\Delta(\hat{\chi}) = \hat{\chi} \otimes 1 + 1 \otimes \hat{\chi} \\ &\Delta(\hat{a}_+) = \hat{a}_+ \otimes 1 + \cosh 2\hat{\chi} \otimes \hat{a}_+ + \sinh 2\hat{\chi} \otimes \hat{a}_+ = \hat{a}_+ \otimes 1 + \mathrm{e}^{2\hat{\chi}} \otimes \hat{a}_+ \\ &\Delta(\hat{a}_-) = \hat{a}_- \otimes 1 + \cosh 2\hat{\chi} \otimes \hat{a}_- - \sinh 2\hat{\chi} \otimes \hat{a}_- = \hat{a}_- \otimes 1 + \mathrm{e}^{-2\hat{\chi}} \otimes \hat{a}_- \end{aligned} \tag{4.7}$$

and the Hopf algebra $Fun_w(ISO(1,1))$ is given by commutation rules (4.1), coproduct (4.7), counit and antipode

$$\epsilon(X) = 0 \qquad X \in \{\hat{a}_+, \hat{a}_-, \hat{\chi}\} \tag{4.8}$$

$$\gamma(\hat{\chi}) = -\hat{\chi} \qquad \gamma(\hat{a}_+) = -\mathrm{e}^{-2\hat{\chi}}\hat{a}_+ \qquad \gamma(\hat{a}_-) = -\mathrm{e}^{2\hat{\chi}}\hat{a}_-. \tag{4.9}$$

Provided that the left and right invariant vector fields obtained from the classical matrix representation similar to (4.6) are

$$X^{\mathrm{L}}_A = \partial_\chi \qquad X^{\mathrm{L}}_{A_+} = \mathrm{e}^{2\chi}\partial_{a_+} \qquad X^{\mathrm{L}}_{A_-} = \mathrm{e}^{-2\chi}\partial_{a_-} \tag{4.10}$$

$$X^{\mathrm{R}}_A = \partial_\chi + 2a_+\partial_{a_+} - 2a_-\partial_{a_-} \qquad X^{\mathrm{R}}_{A_+} = \partial_{a_+} \qquad X^{\mathrm{R}}_{A_-} = \partial_{a_-} \tag{4.11}$$

the Sklyanin bracket induced from $r^{(\mathrm{n})} = A \wedge A_+$ is

$$\{f, g\} = r^{\alpha\beta}(X^{\mathrm{L}}_\alpha f X^{\mathrm{L}}_\beta g - X^{\mathrm{R}}_\alpha f X^{\mathrm{R}}_\beta g) = m \circ ((\mathrm{e}^{2\chi} - 1)\partial_\chi \wedge \partial_{a_+} - 2a_-\partial_{a_+} \wedge \partial_{a_-})(f \otimes g) \tag{4.12}$$

and gives the structure of a Poisson–Hopf algebra to $Fun(ISO(1,1))$.

In particular, for the coordinates a_+, a_- and χ we get

$$\{\chi, a_+\} = \mathrm{e}^{2\chi} - 1 \qquad \{\chi, a_-\} = 0 \qquad \{a_+, a_-\} = -2a_-. \tag{4.13}$$

This means that the commutation rules (4.1) can be seen as a Weyl quantization $\{\,,\,\} \to w^{-1}[\,,\,]$ of the fundamental Poisson brackets given by (4.13) (so have been obtained in [18]) and that coproduct (4.7) does not change under quantization (compare to [4, 12]).

Finally, it is worth recalling that in [4, 5, 12, 13] the T-matrix coordinates that generate $Fun_q(G)$ close a solvable finite dimensional Lie (super) algebra (with the exception of the 'esoteric' quantum $GL(n)$ [5]). In particular, for the $iso(1,1)$ case studied in [12] it is shown that

$$[\pi, \pi_+] = -z\,\pi_+ \qquad [\pi, \pi_-] = -z\,\pi_- \qquad [\pi_+, \pi_-] = 0 \tag{4.14}$$

where $\{\pi, \pi_+, \pi_-\}$ are also quantized light-cone coordinates of $ISO(1,1)$. This is no longer the case for the non-standard deformation (4.1). In fact, if we consider $\hat{a}_+, \hat{a}_-$ and $\hat{f}_m := \mathrm{e}^{2m\hat{\chi}}$ $(m \in \mathbb{Z})$, we can identify $Fun_w(ISO(1,1))$ with an infinite dimensional Lie algebra endowed with a Hopf algebra structure:

$$\begin{aligned}
\Delta(\hat{f}_m) &= \hat{f}_m \otimes \hat{f}_m \\
\Delta(\hat{a}_+) &= \hat{a}_+ \otimes 1 + \hat{f}_1 \otimes \hat{a}_+ \\
\Delta(\hat{a}_-) &= \hat{a}_- \otimes 1 + \hat{f}_{-1} \otimes \hat{a}_-.
\end{aligned} \tag{4.15}$$

Moreover, since

$$[\hat{f}_m, \hat{a}_+] = 2\,w\,m\,(\hat{f}_{m+1} - \hat{f}_m) \tag{4.16}$$

we have an infinite dimensional Hopf subalgebra generated by $\hat{f}_m$ and $\hat{a}_+$ (note that A and A_+ do generate a Hopf subalgebra as well). In principle, this infinite-dimensional aspect makes the approach formally closer to the realistic integrable models obtained without truncating the spectral parameter [5].

5. Applications of the universal $\mathcal{T}$-matrix

5.1. *A change of quantum coordinates*

It would be interesting to compare the non-standard results so far obtained to the standard quantum Poincaré group given in [15], where the group element was

$$G = \begin{pmatrix} 1 & 0 & 0 \\ \hat{a}_1 & \cosh\hat{\theta} & \sinh\hat{\theta} \\ \hat{a}_2 & \sinh\hat{\theta} & \cosh\hat{\theta} \end{pmatrix} \tag{5.1}$$

and the commutation rules between the quantum coordinates read

$$\begin{aligned}
[\hat{\theta}, \hat{a}_1] &= w'\,(\cosh\hat{\theta} - 1) \\
[\hat{\theta}, \hat{a}_2] &= w'\,\sinh\hat{\theta} \\
[\hat{a}_1, \hat{a}_2] &= w'\,\hat{a}_1.
\end{aligned} \tag{5.2}$$

The standard quantum $(1+1)$ Poincaré plane of coordinates $(\hat{x}_1^{(s)}, \hat{x}_2^{(s)})$ characterized by $[\hat{x}_1^{(s)}, \hat{x}_2^{(s)}] = w'\,\hat{x}_1^{(s)}$ is easily derived from these expressions.

To carry out the comparison, the 'light-cone' quantum coordinates $(\hat{a}_+, \hat{a}_-)$ should be transformed into the time and space quantum translations $(\hat{a}_1, \hat{a}_2)$. Both standard and non-standard quantizations turn out to be Weyl quantizations of the classical coordinates preserving the multiplicative property of T^D and G (cf (4.12) and equation (3.1) in [15]). This fact suggests the following relation

$$\hat{\theta} = -2\hat{\chi} \qquad \hat{a}_1 = \hat{a}_- + \hat{a}_+ \qquad \hat{a}_2 = \hat{a}_- - \hat{a}_+. \tag{5.3}$$

This redefinition implies that T^D and G become identical. Hence, we have two different sets of commutation rules compatible with the same coproduct $\Delta(G) = G\dot{\otimes}G$. In particular, the change (5.3) on (4.1) provides the non-standard brackets $(w' = -2w)$

$$\begin{aligned}
[\hat{\theta}, \hat{a}_1] &= w'\,(\cosh\hat{\theta} - 1 - \sinh\hat{\theta}) \\
[\hat{\theta}, \hat{a}_2] &= w'\,(\sinh\hat{\theta} + 1 - \cosh\hat{\theta}) \\
[\hat{a}_1, \hat{a}_2] &= w'\,(\hat{a}_1 + \hat{a}_2).
\end{aligned} \tag{5.4}$$

The non-standard quantum Poincaré plane (whose relations are invariant under the coaction defined by G on a vector $(\hat{x}_1^{(n)}, \hat{x}_2^{(n)})$) is characterized by

$$[\hat{x}_1^{(n)}, \hat{x}_2^{(n)}] = w'(\hat{x}_1^{(n)} + \hat{x}_2^{(n)}). \tag{5.5}$$

A comparison between both quantizations shows that, in this quantum basis, the non-standard deformation seems to be constructed by adding some additional terms on the standard relations. It is also worth remarking the absolutely symmetrical role that both coordinates play in the non-standard case (see the quantum plane relation (5.5)). This kind of 'symmetrical' quantization has already been related to the non-standard $(2+1)$ deformations at a quantum algebra level [19].

In this context, the T-matrix can be used to derive the transformation of the generators of $U_w iso(1,1)$ that is paired by duality to the change of quantum coordinates (5.3). In other words, by using elementary properties of T we can find a set $\{A_1, A_2, A_{12}\}$ of generators of $U_w iso(1,1)$ such that the Hopf algebra dual form T is preserved:

$$T = \mathrm{e}^{A_-\otimes\hat{a}_-}\mathrm{e}^{A_+\otimes\hat{a}_+}\mathrm{e}^{A\otimes\hat{\chi}} = \mathrm{e}^{A_1\otimes\hat{a}_1}\mathrm{e}^{A_2\otimes\hat{a}_2}\mathrm{e}^{A_{12}\otimes\hat{\theta}}. \tag{5.6}$$

Since 'universal' computations to relate both $U_w iso(1,1)$ bases are extremely cumbersome, we can specialize the T-matrix to a (fundamental) representation Q of $Fun_w(ISO(1,1))$ given by

$$Q(\hat{\chi}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Q(\hat{a}_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q(\hat{a}_+) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \\ 0 & 0 & 2w & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

A straightforward computation shows that

$$T^Q := \mathrm{e}^{A_- Q(\hat{a}_-)}\mathrm{e}^{A_+ Q(\hat{a}_+)}\mathrm{e}^{A Q(\hat{\chi})} = \begin{pmatrix} 1 & 0 & A & A_+ \\ 0 & \mathrm{e}^{-2wA_+} & 0 & A_- \\ 0 & 0 & \mathrm{e}^{2wA_+} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.8}$$

From this point of view, $U_w iso(1,1)$ can be seen as a quantum group with non-commuting coordinates A_+, A_- and A. Moreover, the coproduct (3.3) is reproduced by the multiplicative property $\Delta(T^Q) = T^Q\dot{\otimes}T^Q$. In this way, (5.8) could be seen as a transfer matrix for a model with quantum dynamical variables generating a quantum Poincaré algebra (we recall that, for instance, in the sine–Gordon model the dynamical algebra is just $iso(1,1)$ [5]). Therefore, these specializations of universal T-matrices in terms of a representation of a quantum group can be used to construct new models in which the dynamical algebra coincides with the corresponding original quantum enveloping algebra.

The new quantum coordinates admit a representation $Q(\hat{a}_1)$, $Q(\hat{a}_2)$ and $Q(\hat{\theta})$ derived from (5.3) and (5.7). By computing the corresponding exponentials we obtain a second expression for the dual form $T^Q := \mathrm{e}^{A_1 Q(\hat{a}_1)}\mathrm{e}^{A_2 Q(\hat{a}_2)}\mathrm{e}^{A_{12} Q(\hat{\theta})}$:

$$T^Q = \begin{pmatrix} 1 & 0 & -2A_{12} & A_1 - A_2 \\ 0 & \mathrm{e}^{-2w(A_1-A_2)} & 0 & \frac{1}{2w}(\mathrm{e}^{-2w(A_1-A_2)} - 2\mathrm{e}^{-2wA_1} + 1) \\ 0 & 0 & \mathrm{e}^{2w(A_1-A_2)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.9}$$

The relation between the two $U_w iso(1,1)$ bases is now clear

$$A_+ = A_1 - A_2 \qquad A = -2A_{12} \qquad A_- = \frac{1}{2w}(\mathrm{e}^{-2w(A_1-A_2)} - 2\mathrm{e}^{-2wA_1} + 1). \tag{5.10}$$

Note that $\lim_{w\to 0} A_- = A_1 + A_2$ and we recover the usual classical change of basis.

5.2. *The universal R-matrix*

It is also possible to obtain a universal R-matrix defined on a quantum algebra $U_w g$ from the expression of its T-matrix if there exists an algebra homomorphism and coalgebra antihomomorphism Φ that carries the quantum group $Fun_w(G)$ into the quantum algebra $U_w g$. In that case, such a solution of the quantum Yang–Baxter equation (QYBE) would be given by [6]

$$R = (\mathrm{id} \otimes \Phi)\, T \tag{5.11}$$

with Φ operating on the generators of $U_w g$.

It can be checked that for the non-standard quantum Poincaré algebra the homomorphism Φ is given by

$$\Phi(\hat{\chi}) = wA_+ \qquad \Phi(\hat{a}_+) = -wA \qquad \Phi(\hat{a}_-) = 0 \qquad \Phi(1) = 1. \tag{5.12}$$

In fact, the image of $Fun_w(ISO(1,1))$ under Φ is not $U_w iso(1,1)$, but the Hopf subalgebra generated by A and A_+, that we shall denote as $U_w l$; the dual form for $U_w l$ will be

$$T_l = \mathrm{e}^{A_+\otimes \hat{a}_+}\mathrm{e}^{A\otimes\hat{\chi}} \tag{5.13}$$

and $Fun_w(L)$ will be generated by $\hat{\chi}$ and $\hat{a}_+$.

The formula (5.11) applied on (3.7) leads to the following element of $U_w iso(1,1) \otimes U_w iso(1,1)$

$$R = (\mathrm{id} \otimes \Phi)\, T = \mathrm{e}^{-wA_+\otimes A}\mathrm{e}^{wA\otimes A_+} \tag{5.14}$$

that is a triangular solution (i.e., $\sigma \circ R = R^{-1}$, where σ is the flip operator $\sigma(a\otimes b) = b\otimes a$) of the QYBE (this R-matrix is worth comparing with the one deduced in [22]). Moreover, it can be proven that

$$\sigma \circ \Delta(X) = R\,\Delta(X)\,R^{-1} \qquad \text{for } X = \{A, A_+, A_-\}. \tag{5.15}$$

In particular, it is interesting to show explicitly how the relation (5.15) is fulfilled in the case of A_-. Let us consider the new generator $\tilde{A_-} = A_-\mathrm{e}^{2wA_+}$, whose commutation rules with A, A_+ and coproduct are

$$[A, \tilde{A_-}] = -2\tilde{A_-} \qquad [A_+, \tilde{A_-}] = 0 \qquad \Delta(\tilde{A_-}) = 1\otimes\tilde{A_-} + \tilde{A_-}\otimes\mathrm{e}^{2wA_+}. \tag{5.16}$$

To compute the right-hand side of (5.15), we take into account that

$$\mathrm{e}^{f}\,\Delta(\tilde{A_-})\mathrm{e}^{-f} = \Delta(\tilde{A_-}) + \sum_{n=1}^{\infty}\frac{1}{n!}[f, \ldots [f, \Delta(\tilde{A_-})]^{n)} \ldots]. \tag{5.17}$$

Setting $f = wA\otimes A_+$ we find that ($n \geqslant 1$)

$$[wA\otimes A_+, \ldots [wA\otimes A_+, \Delta(\tilde{A_-})]^{n)} \ldots] = (-2w)^n\tilde{A_-}\otimes A_+^n\mathrm{e}^{2wA_+} \tag{5.18}$$

and, in this way, we get

$$\begin{aligned}\mathrm{e}^{wA\otimes A_+}\,\Delta(\tilde{A_-})\,\mathrm{e}^{-wA\otimes A_+} &= \Delta(\tilde{A_-}) + \sum_{n=1}^{\infty}\frac{(-2w)^n}{n!}\tilde{A_-}\otimes A_+^n\mathrm{e}^{2wA_+}\\ &= \Delta(\tilde{A_-}) + \tilde{A_-}\otimes(\mathrm{e}^{-2wA_+} - 1)\mathrm{e}^{2wA_+} = 1\otimes\tilde{A_-} + \tilde{A_-}\otimes 1 = \Delta_0(\tilde{A_-}).\end{aligned} \tag{5.19}$$

Note that $e^{wA\otimes A_+}$ turns $\Delta(\tilde{A_-})$ into $\Delta_0(\tilde{A_-})$. Since

$$[-wA_+\otimes A,\ldots[-wA_+\otimes A,\Delta_0(\tilde{A_-})]^{n)}\ldots]=(2w)^n A_+^n\otimes\tilde{A_-} \tag{5.20}$$

the passage from $\Delta_0(\tilde{A_-})$ to $\sigma\circ\Delta(\tilde{A_-})$ is given by $e^{-wA_+\otimes A}$:

$$\begin{aligned} e^{-wA_+\otimes A}\,\Delta_0(\tilde{A_-})\,e^{wA_+\otimes A} &= \Delta_0(\tilde{A_-})+\sum_{n=1}^{\infty}\frac{(2w)^n}{n!}A_+^n\otimes\tilde{A_-}\\ &= \Delta_0(\tilde{A_-})+(e^{2wA_+}-1)\otimes\tilde{A_-}=\tilde{A_-}\otimes 1+e^{2wA_+}\otimes\tilde{A_-}=\sigma\circ\Delta(\tilde{A_-}). \end{aligned} \tag{5.21}$$

Finally, the condition (5.15) for A_- is straightforwardly obtained from that of $\tilde{A_-}$.

The (quadratic) commutation rules between the entries of the quantum matrix (4.6) that are derived from (4.1) coincide with the relations obtained via the FRT [23] prescription and with the aid of a particular representation (2.7) of the quantum R-matrix

$$R=I\otimes I+wD(A)\wedge D(A_+) \tag{5.22}$$

where I is the 3×3 identity matrix.

Similarly, it can be shown that duality arguments imply that

$$\hat{R}=\exp\left\{\frac{1}{w}\,\hat{a}_+\otimes\hat{\chi}\right\}\exp\left\{\frac{-1}{w}\,\hat{\chi}\otimes\hat{a}_+\right\} \tag{5.23}$$

verifies both QYBE and (5.15) for $Fun_w(ISO(1,1))$.

5.3. Quantum (1 + 1) Poincaré and Galilei with dilations

Let us consider two copies $U_w l^1$ and $U_{-w} l^2$ (with generators A^1, A^1_+ and A^2, A^2_+) of the (quantum) Hopf subalgebra $U_w l$ of $U_w iso(1,1)$. It can be easily checked that the quantum algebra $U_w l^1\oplus U_{-w} l^2$ is a triangular deformation $U_w\bar{p}$ of a $(1+1)$ Poincaré algebra enlarged with a spacetime dilation. Explicitly, by introducing the boost K, the time and space translations H, P and the dilation D as

$$\begin{aligned} &D=\tfrac{1}{2}(A^1+A^2) \qquad K=\tfrac{1}{2}(A^1-A^2)\\ &H=A^1_++A^2_+ \qquad P=A^1_+-A^2_+ \end{aligned} \tag{5.24}$$

the following coproduct and (non-vanishing) commutation rules are derived:

$$\begin{aligned} &\Delta(H)=H\otimes 1+1\otimes H \qquad \Delta(P)=P\otimes 1+1\otimes P\\ &\Delta(D)=1\otimes D+D\otimes e^{wP}\cosh(wH)+K\otimes e^{wP}\sinh(wH)\\ &\Delta(K)=1\otimes K+K\otimes e^{wP}\cosh(wH)+D\otimes e^{wP}\sinh(wH) \end{aligned} \tag{5.25}$$

$$\begin{aligned} &[K,H]=\frac{1}{w}\,(e^{wP}\cosh(wH)-1) \qquad [K,P]=\frac{1}{w}\,e^{wP}\sinh(wH)\\ &[D,H]=\frac{1}{w}\,e^{wP}\sinh(wH) \qquad [D,P]=\frac{1}{w}\,(e^{wP}\cosh(wH)-1). \end{aligned} \tag{5.26}$$

The classical r-matrix underlying this deformation is $r=r^1-r^2=w(D\wedge P+K\wedge H)$, and the universal T-matrix linked to $U_w\bar{p}$ will be the product of two T_l dual forms (5.13):

$$\mathcal{T}=T_l{}^1\,T_l{}^2=e^{A^1_+\otimes\hat{a}^1_+}\,e^{A^1\otimes\hat{\chi}^1}\,e^{A^2_+\otimes\hat{a}^2_+}\,e^{A^2\otimes\hat{\chi}^2}. \tag{5.27}$$

Under the change of basis (5.24), this element can be written as

$$\mathcal{T}=e^{H\otimes\hat{h}}\,e^{P\otimes\hat{p}}\,e^{D\otimes\hat{d}}\,e^{K\otimes\hat{k}} \tag{5.28}$$

where the change of quantum coordinates is given by

$$\hat{d}=\hat{\chi}^1+\hat{\chi}^2 \qquad \hat{h}=\tfrac{1}{2}(\hat{a}^1_++\hat{a}^2_+)$$

$$\hat{k} = \hat{\chi}^1 - \hat{\chi}^2 \qquad \hat{p} = \tfrac{1}{2}(\hat{a}_+^1 - \hat{a}_+^2). \tag{5.29}$$

As a consequence, the corresponding quantum Poincaré group $Fun_w(\bar{P})$ will be

$$\begin{aligned} &[\hat{k}, \hat{h}] = w\,(\mathrm{e}^{\hat{d}}\cosh(\hat{k}) - 1) \qquad [\hat{k}, \hat{p}] = w\,\mathrm{e}^{\hat{d}}\sinh(\hat{k}) \\ &[\hat{d}, \hat{h}] = w\,\mathrm{e}^{\hat{d}}\sinh(\hat{k}) \qquad [\hat{d}, \hat{p}] = w\,(\mathrm{e}^{\hat{d}}\cosh(\hat{k}) - 1) \end{aligned} \tag{5.30}$$

$$\begin{aligned} &\Delta(\hat{k}) = \hat{k} \otimes 1 + 1 \otimes \hat{k} \qquad \Delta(\hat{d}) = \hat{d} \otimes 1 + 1 \otimes \hat{d} \\ &\Delta(\hat{h}) = \hat{h} \otimes 1 + \mathrm{e}^{\hat{d}}\cosh(\hat{k}) \otimes \hat{h} + \mathrm{e}^{\hat{d}}\sinh(\hat{k}) \otimes \hat{p} \\ &\Delta(\hat{p}) = \hat{p} \otimes 1 + \mathrm{e}^{\hat{d}}\cosh(\hat{k}) \otimes \hat{p} + \mathrm{e}^{\hat{d}}\sinh(\hat{k}) \otimes \hat{h}. \end{aligned} \tag{5.31}$$

An isomorphism between the two algebras (5.26) and (5.30) (which is a coalgebra anti-isomorphism) can be easily found starting by (5.12) and taking into account (5.24). It reads

$$\begin{aligned} &\Phi(\hat{k}) = w\,H \qquad \Phi(\hat{d}) = w\,P \qquad \Phi(\hat{h}) = -w\,K \\ &\Phi(\hat{p}) = -w\,D \qquad \Phi(1) = 1. \end{aligned} \tag{5.32}$$

Hence, the universal R-matrix for $U_w\bar{p}$ will be

$$\mathcal{R} = \mathrm{e}^{-w\,H\otimes K}\,\mathrm{e}^{-w\,P\otimes D}\,\mathrm{e}^{w\,D\otimes P}\,\mathrm{e}^{w\,K\otimes H} \tag{5.33}$$

a result that is worth comparing with the one obtained in [22] from a different method. Note that, by construction, the R-matrix (5.33) fulfils the equation (5.15) for all the generators of $U_w\bar{p}$.

The non-relativistic limit of this deformation can be easily computed by means of a contraction. The contracted generators of the quantum (1 + 1) Galilei algebra $U_w\bar{g}$ are defined in terms of the contraction mapping

$$K' = \varepsilon\,K \qquad H' = H \qquad P' = \varepsilon\,P \qquad D' = D. \tag{5.34}$$

The associated quantum contracted coordinates will be (cf [12])

$$\hat{k}' = \varepsilon^{-1}\,\hat{k} \qquad \hat{h}' = \hat{h} \qquad \hat{p}' = \varepsilon^{-1}\,\hat{p} \qquad \hat{d}' = \hat{d}. \tag{5.35}$$

If we define the contracted deformation parameter as $w' = \varepsilon^{-1}\,w$ and compute the limit $\varepsilon \to 0$, we obtain that $U_{w'}\bar{g}$ will be given by

$$\begin{aligned} &\Delta(H') = H' \otimes 1 + 1 \otimes H' \qquad \Delta(P') = P' \otimes 1 + 1 \otimes P' \\ &\Delta(D') = 1 \otimes D' + D' \otimes \mathrm{e}^{w'P'} + w'\,K' \otimes \mathrm{e}^{w'P'}\,H' \\ &\Delta(K') = 1 \otimes K' + K' \otimes \mathrm{e}^{w'P'} \end{aligned} \tag{5.36}$$

$$[K', H'] = \frac{1}{w'}\,(\mathrm{e}^{w'P'} - 1) \qquad [D', H'] = \mathrm{e}^{w'P'}\,H' \qquad [D', P'] = \frac{1}{w'}\,(\mathrm{e}^{w'P'} - 1). \tag{5.37}$$

Note that the r-matrix for $U_{w'}\bar{g}$ is $r' = w'(D' \wedge P' + K' \wedge H')$, and the contracted universal T-matrix reads

$$\mathcal{T}' = \mathrm{e}^{H'\otimes\hat{h}'}\,\mathrm{e}^{P'\otimes\hat{p}'}\,\mathrm{e}^{D'\otimes\hat{d}'}\,\mathrm{e}^{K'\otimes\hat{k}'}. \tag{5.38}$$

On the other hand, the contraction (5.35) applied onto the quantum Poincaré group (5.30)–(5.31) gives rise to the following quantum Galilei group $Fun_{w'}(\bar{G})$:

$$[\hat{k}', \hat{h}'] = w'\,(\mathrm{e}^{\hat{d}'} - 1) \qquad [\hat{k}', \hat{p}'] = w'\,\mathrm{e}^{\hat{d}'}\hat{k}' \qquad [\hat{d}', \hat{p}'] = w'\,(\mathrm{e}^{\hat{d}'} - 1) \tag{5.39}$$

$$\begin{aligned} &\Delta(\hat{k}') = \hat{k}' \otimes 1 + 1 \otimes \hat{k}' \qquad \Delta(\hat{d}') = \hat{d}' \otimes 1 + 1 \otimes \hat{d}' \\ &\Delta(\hat{h}') = \hat{h}' \otimes 1 + \mathrm{e}^{\hat{d}'} \otimes \hat{h}' \qquad \Delta(\hat{p}') = \hat{p}' \otimes 1 + \mathrm{e}^{\hat{d}'} \otimes \hat{p}' + \mathrm{e}^{\hat{d}'}\hat{k}' \otimes \hat{h}' \end{aligned} \tag{5.40}$$

and the algebra isomorphism and coalgebra anti-isomorphism Φ' between $Fun_{w'}(\bar{G})$ and $U_{w'}\bar{g}$ is now

$$\begin{aligned} &\Phi'(\hat{k}') = w'\, H' \qquad \Phi'(\hat{a}') = w'\, P' \qquad \Phi'(\hat{h}') = -w'\, K' \\ &\Phi'(\hat{p}') = -w'\, D' \qquad \Phi'(1) = 1. \end{aligned} \tag{5.41}$$

Hence, the universal R-matrix for $U_{w'}\bar{g}$ is written in the form

$$\mathcal{R}' = \mathrm{e}^{-w'\, H'\otimes K'}\, \mathrm{e}^{-w'\, P'\otimes D'}\, \mathrm{e}^{w'\, D'\otimes P'}\, \mathrm{e}^{w'\, K'\otimes H'}. \tag{5.42}$$

This result is quite interesting since the algebra $U_{w'}\bar{g}$ is the null-plane isotopy (Hopf) subalgebra included in the null-plane deformation of the $(2+1)$ Poincaré algebra [19]. Hence, (5.42) is a quantum R-matrix for this non-standard Poincaré deformation, although the equation (5.15) is, in principle, not fulfilled for the two remaining Poincaré generators out of this subalgebra (the so-called quantum Hamiltonians [24]).

6. Concluding remarks

Non-standard quantum deformations have received scant attention compared to the standard ones. However, they present interesting features: the existence of a $*_\hbar$-product that quantizes the Poisson–Lie group is always guaranteed for them, and they are naturally adapted to the null-plane basis of the Poincaré algebra (see [19, 24] for the construction of such null-plane deformations of the $(2+1)$ and $(3+1)$ dimensional cases).

It is interesting to note that the essential features of the T-matrix approach have been obtained without computing all the components of the dual tensors. We have also shown that the quantum group $Fun_w(ISO(1,1))$ is, in fact, an infinite dimensional Hopf–Lie algebra. This structure is rather different to the universal enveloping algebras encountered when computing T-matrices in the literature.

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